Ferrite Resonance Isolators Using Tapered DC Magnetic Fields*

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Summary—Tapered dc magnetic fields can be used to broad-band ferrite resonance isolators of the rectangular waveguide type. In this work a ferrite slab of narrow resonance line width was employed in order to keep the ratio of reverse-to-forward attenuation large while the bandwidth of the isolator was broadened by means of an appropriately tapered magnetic field. A ratio of reverse-to-forward attenuation of more than 60 has been obtained experimentally over a 38 per cent frequency range at *X* band, and further improvement of these figures seems possible.

I. Introduction

IT HAS BEEN shown by Schlömann[1] that the bandwidth of ferrite resonance isolators in rectangular waveguides is proportional to, but smaller than, the width of the resonance line of the ferrite. He also shows that the maximum attainable ratio of reverse-to-forward attenuation is inversely proportional to the square of the linewidth. Thus it would appear that the requirements on the linewidth of ferrite for broad-band isolators and for a high ratio of reverse-to-forward attenuation are incompatible. In this paper it will be shown that this apparent limitation does not exclude the simultaneous realization of both large bandwidth and large reverse-to-forward attenuation ratio. A tapered magnetic field having a predetermined transverse taper serves to broad band the isolator. At the same time a high ratio of reverse-to-forward attenuation is maintained by using a ferrite slab of narrow linewidth.

The idea of using tapered dc magnetic fields in connection with ferrite resonance isolators is not new. In a broad sense isolators could be considered here having a longitudinally tapered or stepped magnetic field. Similarly, isolators could be included having a longitudinal taper or step with respect to saturation magnetization of the ferrite loading. Theoretically, such isolators might be considered as a number of narrow-band isolators in series. The broader band is simply bought at the expense of structure length and to some extent of the reverse-to-forward ratio. In this sense the theory of longitudinally tapered isolators is straightforward and will not be considered here. Thus only isolators with a transversely tapered field will be treated in this paper. Soohoo[2] also discussed the resonance isolator with a transversely tapered dc field of the distribution different from the one to be described here.

In general, the reasons for using a transversely tapered field for a resonance isolator may be classified into three categories. The first is to extend the bandwidth of the isolator. Any transversely tapered field will accomplish this purpose since, in effect, it broadens the linewidth of the material. In this fashion it is easy to broaden the frequency range over which high reverse attenuation is observed. Usually, however, a decrease of the ratio of reverse-to-forward attenuation results from this broadening. This is in accordance with Schlömann's calculation. that a ferrite material with large linewidth produces a smaller ratio of reverse-to-forward attenuation. In order to keep the forward loss within smaller limits, it is, therefore, necessary to keep the RF field distribution more nearly constant with frequency change. Means to this end were devised and they resulted indeed in isolators having desirable broad-band characteristics. The techniques used here include partial dielectric loading of the waveguide[3–5] or design of a special transmission structure.[6]

A second application of transversely tapered dc fields has led to the development of isolators having an increased ratio of reverse-to-forward attenuation over a narrow frequency band. Apparently, the reverse attenuation obtained with tapered field is somewhat smaller than that with homogeneous field. It can be shown, however, that the forward attenuation is reduced much more drastically by the tapered field. This is so because the regions of the ferrite which normally give rise to most of the forward loss are now tuned off resonance. This explanation[7] accounts for the high ratio of the so-called "Blasburg isolator."

A third way of using a transversely tapered dc field in a ferrite resonance isolator is treated in this paper. The objective is to combine the advantages of both cases discussed before, *i.e.*, a broad band and high

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[6] W. W. Anderson and M. E. Hines, "Wide-band resonance isolator," IRE Trans. on Microwave Theory and Techniques, vol. MTT-9, pp. 63–67; January, 1961.

[7] B. Lax, "Frequency and loss characteristics of microwave ferrite devices," Proc. IRE, vol. 44, pp. 1368–1386; October, 1956.

reverse-to-forward attenuation ratio. One may ask whether it is possible to achieve both goals in a conventional resonance isolator structure simply by applying a tapered dc field, and if so, what will be the optimum distribution of the dc field.

Consider a thin slab of ferrite placed inside a rectangular waveguide as shown in Fig. 1. Assume the ferrite slab is thin enough so that perturbation theory holds. Then the RF magnetic field inside the ferrite slab is the same as the RF magnetic field of the empty waveguide at the site of the slab. Suppose the dc magnetic field normal to the ferrite slab at the portion of the ferrite "seeing" the circularly polarized RF field has the proper magnitude for ferromagnetic resonance. This will provide a large absorption by the ferrite of a wave propagating in one direction. However, for a wave propagating in the opposite direction the RF magnetic field at this position has the opposite sense of circular polarization and thus the absorption by this portion of the ferrite can be kept small. The contribution to the forward attenuation from the other parts of the ferrite slab is small since the dc field is tuned off resonance there. If the dc magnetic field is so shaped that the same conditions hold as the position of circular polarization moves with frequency, one should be able to construct an isolator with a bandwidth covering essentially the entire frequency band of the dominant waveguide mode.

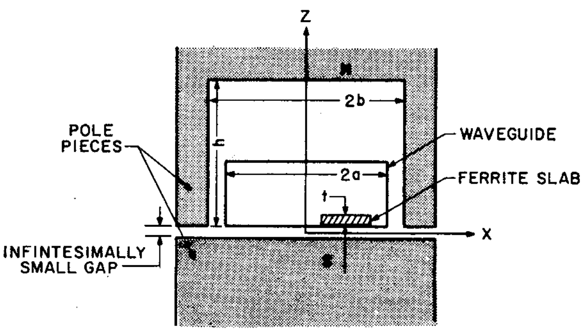

Fig. 1—The ferrite resonance isolator using a tapered dc field.

II. Analysis of the Isolator

It will be assumed that each part of the ferrite slab responds independently to the applied magnetic field and the demagnetizing field of the sample (independent grain hypothesis). Each grain has a resonance frequency different from the others, since the applied dc field is tapered. The absorption of the isolator is taken to be the sum of the absorption of the individual grain of the ferrite. The only interaction between the grains retained in this hypothesis is the demagnetizing field due to the poles which appear on the surface of the sample.

When the dc field normal to the plane of the slab changes its magnitude with position and only part of the ferrite slab engages in ferromagnetic resonance, magnetic poles appear on the boundary between the parts in resonance and not in resonance. The effect of these magnetic poles may be expressed effectively by N_x in the calculation of the attenuation constants of the isolator. A word of caution is in place here with reference to using the transverse or RF demagnetizing factor N_x. Strictly speaking, such a demagnetizing factor is defined only for a situation where the entire sample resonates at one frequency. However, since N_x would be small for the flat slab geometry used, the error introduced by an indiscriminate use of N_x cannot be serious. In fact, in most calculations in this work N_x was assumed to be zero, although a correction term including small N_x can be calculated without much difficulty.

The perturbation theory[7] will be used to calculate the attenuation constants for the reverse and forward directions[8] α_+ and α_-, respectively. The theory assumes that the energy in the ferrite is much smaller than that in the empty waveguide. Since the ferrite slab is thin and only part of the slab is in ferromagnetic resonance for a given frequency, the perturbation theory should be appropriate here.

The coordinates are designed as in Fig. 1, and the ferrite slab is assumed to lie only in the region $x>0$. Then α_+ and α_- can be expressed as

$$\alpha_+ = C\iint [h_+^*\chi_{++}''h_+ + h_-^*\chi_{--}''h_- + h_+^*\chi_{+-}''h_- + h_-^*\chi_{-+}''h_+]dxdz \quad (1)$$

$$\alpha_- = C\iint [h_-^*\chi_{++}''h_- + h_+^*\chi_{--}''h_+ + h_-^*\chi_{+-}''h_+ + h_+^*\chi_{-+}''h_-]dxdz \quad (2)$$

where C is the constant of proportionality and it is a function of the cross-sectional area of the waveguide. The integration is to be taken over the cross section of the ferrite. h_{+-} are the positive and negative circular components of the RF magnetic field at the site of the ferrite, respectively. χ'''s are the imaginary parts of the components of the external susceptibility tensor. The asterisk* designates the complex conjugate. The greater symmetry of a Polder matrix results if the demagnetizing factors in the x and y directions are equal, that is, if $N_x=N_y$. Then $\chi_{+-}''=0$, and (1) and (2) become

$$\alpha_+ = C\iint [|h_+|^2\chi_{++}'' + |h_-|^2\chi_{--}'']dxdz \quad (3)$$

$$\alpha_- = C\iint [|h_-|^2\chi_{++}'' + |h_+|^2\chi_{--}'']dxdz. \quad (4)$$

χ_{++}'' has a resonance peak at the Kittel's resonance frequency,[9] while χ_{--}'' is a very small quantity without resonance peak. Note that χ'''s are functions of x

[8] The power attenuation by passing through a section of length l is 8.68 $\alpha_+ l$ db.

[9] C. Kittel, "On the theory of ferromagnetic resonance absorption," *Phys. Rev.*, vol. 73, pp. 155–161; January, 1948.

since the dc magnetic field normal to the ferrite slab changes its magnitude along the x axis. For $x>0$, h_+ varies slowly with x, but h_- becomes zero at a particular value of x. Thus one sees that the peak of χ_{++}'' and the valley of h_- should nearly coincide in order to reduce α_-. The other three terms in (3) and (4) are relatively insensitive to the variation of h's and χ'''s along the x axis.

Consider the dc magnetic field. Since the ferrite is placed "infinitesimally" close to the magnet pole piece, it is exposed to a field having only an H_z component, not H_x. A function $g(\theta)$ is introduced which describes the distribution of H_z along the x axis at $z=0$,

$$H_z = H_{z0}g(\theta) \tag{5}$$

with

$$\theta = \frac{\pi x}{2a}, \qquad \theta = 0 \cdots \frac{\pi}{2}, \tag{6}$$

where H_{z0} is the value of H_z at $x=0$, $z=0$ and $2a$ is the width of the waveguide.

From the magnetic field distribution $g(\theta)$, a distribution function $f(\theta)$ for the ferrimagnetic resonance frequency can be found

$$\omega = \omega_0 f(\theta). \tag{7}$$

Here ω_0 is the frequency corresponding to the effective field[10] at $z=0$. The relation between ω and H_z or $f(\theta)$ and $g(\theta)$ is given by Kittel's resonance condition.[9] This can be approximated for a flat slab geometry by

$$\omega \doteq \gamma H_z - \gamma\left(N_z - \frac{N_x}{2}\right)4\pi M. \tag{8}$$

The approximation is valid for

$$N_x \ll \left(\frac{H_z}{4\pi M}\right) - N_z. \tag{9}$$

Here γ is the gyromagnetic ratio ($\gamma/2\pi = 2.8$ Mc/oe for most ferrite materials), N_x and N_z are the demagnetizing factors in the x and z directions, respectively, and $4\pi M$ is the saturation magnetization. Thus

$$f(\theta) = \frac{\gamma}{\omega_0}\left[H_{0z}g(\theta) - \left(N_z - \frac{N_x}{2}\right)4\pi M\right]. \tag{10}$$

The objective of the next step in the theoretical treatment now should be to determine an optimum function $f(\theta)$; this might include the following steps. From an arbitrary $f(\theta)$, one finds the components of the χ'' matrix as functions of x at any frequency ω. The integrals (3) and (4) then give the reverse and forward attenuation at any frequency ω. By varying the originally arbitrary function $f(\theta)$, an optimum $f(\theta)$ should be found which makes the ratio α_+/α_- as high as possible over as wide a frequency band as possible. It is clear that this procedure would be very cumbersome, involving either variational or integral equation calculus.

Instead, a simpler approach can be taken; it will lead to a distribution $f(\theta)$ resulting in very nearly the theoretical optimum ratio α_+/α_- over a broad band. This approach is based on the distribution of the circular magnetic RF field components. At any frequency ω in the band, there is a position x or θ where the negative circular component vanishes, $h_-=0$. It is given by

$$\sec\theta = \frac{\omega}{\omega_c} \tag{11}$$

where ω_c is the angular cutoff frequency of the guide. For frequencies in the conventionally utilized part of the waveguide band, h_+ is not very far from its maximum value at the point where $h_-=0$.

This suggests the following procedure. For any frequency ω, the field is tuned to resonance at the position x or θ where $h_-=0$. From (7) and (11), this condition will be met if

$$\omega_0 = \omega_c \tag{12}$$

and

$$f(\theta) = \sec\theta. \tag{13}$$

Thus the peak of χ_{++}'' and the valley of h_- coincide. The ferrite material located at x contributes nothing to the term $|h_-|^2\chi_{++}''$ of the forward loss, and an appreciable amount although not quite the maximum to the reverse loss, $|h_+|^2\chi_{++}''$. Ferrite material at locations slightly right or left of x, however, will contribute to $|h_-|^2\chi_{++}''$ of the forward loss. This part of the contribution to the forward loss is much smaller than in the case of a uniform dc field since χ_{++}'' is no longer at its peak value at this part of the ferrite slab. The term $|h_+|^2\chi_{--}''$ of the forward attenuation is nearly independent of the choice of the dc field distribution. It can be reduced by using the ferrite material of narrow linewidth. Thus the procedure expressed by (12) and (13) gives nearly the optimum ratio over a wide frequency band.

Combining (10), (12) and (13), the dc field along the x axis should be

$$H_z = H_{e0}\sec\frac{\pi x}{2a} + \left(N_z - \frac{N_x}{2}\right)4\pi M \tag{14}$$

where H_{e0} is the effective field value needed to produce resonance at the cutoff frequency ω_c, *i.e.*, $H_{e0}=\omega_c/\gamma$.

By means of the Schwartz-Christoffel transformation, it can be shown that the pole pieces shown in Fig. 1 produce a field H_z along the x axis approximately as

$$H_z \doteq H_{z0}\sec\left(\frac{\pi x}{2b}\right), \qquad (h \geq 2b) \tag{15}$$

[10] The effective field is the external apply field minus the demagnetizing field.

for a small range of x/a, say $\frac{1}{3} < x/a < \frac{3}{4}$, where the null of the circular field h_- occurs for the frequency range of interest. Eq. (15) becomes exact if h in Fig. 1 goes to infinity and if the gap spacing near the edge becomes truly infinitesimal. For this range of x/a, one can show that the dc field of (15) approximately meets the required field of (14) by proper adjustment of b/a. Also the applied dc field at $x=0$, $z=0$ should be

$$H_{z0} = H_{e0} + \left(N_z - \frac{N_x}{2}\right) 4\pi M. \tag{16}$$

Eqs. (3) and (4) are evaluated as follows. For every frequency $\omega = \omega_1$ the integrand is then expanded in Taylor series about $\theta = \theta_1$ where $|h_-| = 0$ at ω_1. The integral is then carried out for the first term of the expansion, and the result is plotted in Fig. 2 as a function of frequency. The outline of the calculation is given in the Appendix. In Fig. 2

t = thickness of the ferrite slab
S = area of the cross section of the waveguide
$\omega_m = \gamma 4\pi M$
$T = 2/\gamma \Delta H$

where ΔH is the ferromagnetic resonance linewidth and θ_1 can be converted to the frequency of the wave by the relation $\omega_1 = \omega_c \sec \theta_1$. A frequency scale applicable to X-band waveguide is also shown. It has been assumed that $N_y = N_x = 0$ in Fig. 2.

The forward attenuation α_- is generally proportional to the linewidth of the ferrite, but in this broad-band isolator the reverse attenuation α_+ has a much smaller dependence on the linewidth. This can be seen from the observation that α_+ is the integration of the curves of χ_{++}'' vs dc field weighted for the RF distribution in the tapered dc field. For resonance isolators with uniform dc fields, however, α_+ would be approximately inversely proportional to the linewidth since α_+ is proportional to the peak of the resonance curve of the ferrite.

When the ferrite slab extends only from $x = a/3$ to $2a/3$, the dependence of α_+ on the linewidth becomes more significant near the lower frequency ends of the band. (See the solid curves in Fig. 2.) This is because the limit of integration in calculating α_+ is from $x = a/3$ to $2a/3$ and does not cover the whole area under the curve of χ_{++}'' vs dc field of the ferrite. The larger the linewidth the flatter the χ_{++}'' curve, and less area of χ_{++}'' curve is included in the integration, resulting in smaller α_+. At the higher frequency ends of the band, the dc field changes so rapidly with x that still the major fraction of the χ_{++}'' curve lies inside the integration limit of α_+. Thus α_+ depends less on the linewidth of the ferrite at the higher frequency ends of the band.

Both of α_+ and α_- are large at lower frequencies since the dc field varies slowly in this region and a larger part of the ferrite slab participates in the ferromagnetic resonance. Since α_+ and α_- are both proportional to the thickness of the ferrite slab, t, one may vary t as a function of x to make α_+ and α_- more nearly constant over the frequency band. One also notes from Fig. 2 that α_- increases rapidly with the total width of the slab while α_+ is relatively independent of it. Thus for higher $R = \alpha_+/\alpha_-$, the width of the slab should only cover the range of x where circular polarization exists.

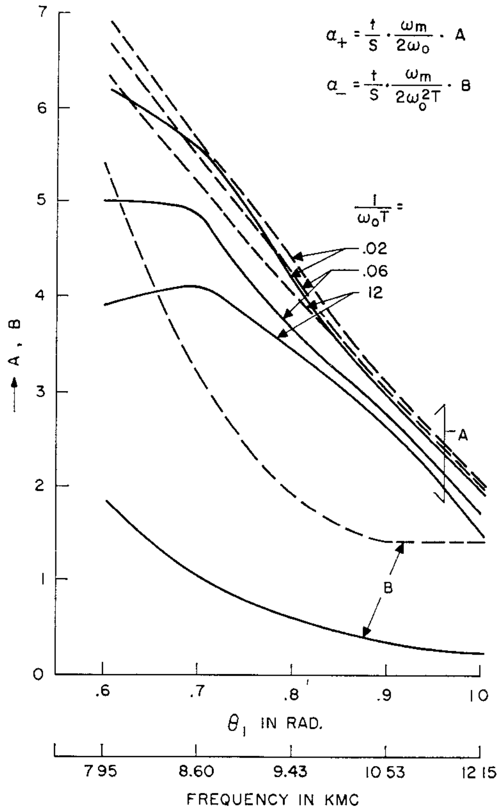

Fig. 2—Constants A and B vs θ_1. The reverse and forward attenuation constants can be expressed as

$$\alpha_+ = \frac{t}{S}\cdot\frac{\omega_m}{2\omega_0}\cdot A$$

and

$$\alpha_- = \frac{t}{S}\cdot\frac{\omega_m}{2\omega_0^2 T}\cdot B,$$

respectively. Frequency f is related to θ_1 by $f = f_c \sec \theta_1$ where f_c is the cutoff frequency of the waveguide. Solid curves are for the ferrite slab covering from $x/a = \frac{1}{3}$ to $\frac{2}{3}$. Dashed lines are for the slab covering from $x/a = 0$ to 1. It is assumed that $N_x = N_y = 0$.

III. Experiments

Pole pieces having a cross section similar to Fig. 1 with $h = 2$ in and $2b = 1$ in were made. Except near the edge of the pole pieces and the ferrite slab, the dc field distribution agreed well with that of the calculation. The ferrite slab was assumed to be thin enough to justify the use of perturbation theory and to minimize the disturbance of the dc magnetic field. Two slabs of polycrystalline YIG 5 in long, 0.02 in thick and 0.15 in and 0.45 in wide (covering from $x = a/3$ to $2a/3$ and $x = 0$ to a, respectively) were made. The slab was put inside a half-height X-band waveguide in turn, and

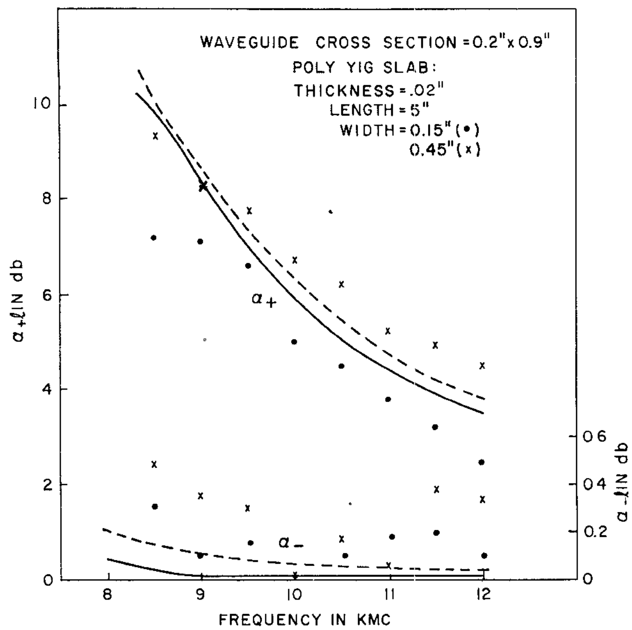

Fig. 3—Measured and calculated forward and reverse attenuation of the isolator using polycrystalline YIG. Solid and dashed curves are the calculated attenuation for the slab width 0.15 in ($x=a/3$ to $2a/3$) and 0.45 in ($x=0$ to a), respectively. It is assumed that $\Delta H=100$ oe, $4\pi M=1750$ oe and $N_x=0$.

α_+ and α_- were measured as a function of frequency. The dc magnetic field was adjusted for the smallest α_- over the widest possible frequency band. These measurements were compared with the calculation. Fig. 3 shows the results. One sees that α_+ agrees fairly well with the analysis while α_- shows higher values. This might be expected since α_- is more sensitive than α_+ to the deviation of the distribution of dc magnetic fields from the optimum distribution assumed in the analysis. The dielectric loss may also contribute to this higher value of the observed α_-. The deviation of the measured α_+ from the calculation near the lower frequencies for the 0.15-in slab could be due to an inaccuracy in positioning the ferrite slab inside the waveguide. It was assumed in calculating the theoretical curves in Fig. 3 that $N_x=0$, $\Delta H=100$ oe and $4\pi M=1750$ gauss.

It has been mentioned in the previous section that the thickness of the slab can be varied along the x direction so that α_+ and α_- are more nearly uniform over the frequency band. The same material of 5 in long and the cross section shown in Fig. 4 was made, and the results are shown in the same figure. The thickness of the slab increases linearly from $x=0.15$ in to 0.3 in where the position of the circular polarization exists. This should make α_+ fairly uniform over the bandwidth. Outside of this range of x, the thickness of the slab is gradually reduced in order to minimize the distortion of the dc magnetic field. The ratio of α_+/α_- is over 60 from 8.2 kMc to 12 kMc. Also in Fig. 4, measurements made on a ferrite slab of the same dimensions but of a different material (XMC-1400) are shown. This material has $4\pi M=1360$ gauss and $\Delta H \doteq 300$ oe. From the analysis one expects a smaller

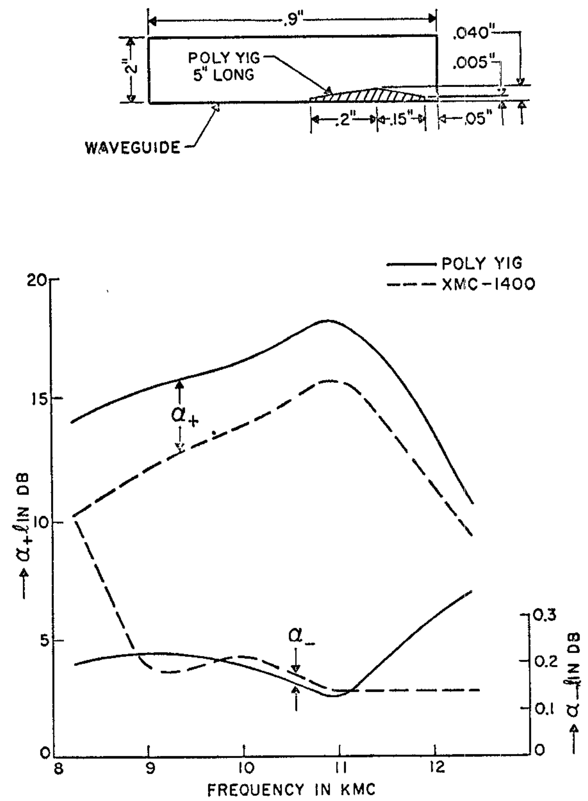

Fig. 4—Measured forward and reverse loss of an isolator using a ferrite slab of the cross section shown.

α_+ for XMC-1400 since ΔH is larger and also $4\pi M$ is smaller than that of poly YIG. This is noticed from the curves in Fig. 4. Also, one notices that the difference in α_+ with the two materials of different linewidth is larger at the lower frequencies than at the higher frequencies. This agrees qualitatively with the calculation shown in Fig. 2. One expects the ratio of α_- for these two materials to be around 2.3 from the analysis, but the measurements did not reflect this expectation. Presumably, this is due to a nonoptimum distribution of dc magnetic field, nonuniformity in the material and dielectric losses.

IV. Conclusions

In the experiments a ratio of reverse-to-forward attenuation in excess of 60 was obtained over a 40 per cent bandwidth. This matches the performance of the best commercial X-band resonance isolators. It falls short, on the other hand, of the theoretical predictions; Fig. 2 shows that a ratio greater than 150 should be expected over a 40 per cent band if a ferrite of a 100-oe (or smaller) linewidth is employed. Thus it appears that there is the possibility of further improvements with respect to the reverse-to-forward attenuation ratio. This would primarily involve some empirical optimization of the shape of the magnet pole pieces, of the ferrite cross section and possibly the waveguide geometry. In addition, a higher ratio can be achieved if isolation is required over a narrower band.

One disadvantage of the isolator in its present form is the comparatively low reverse loss-per-unit length. This can be explained on two grounds. Whereas other isolators employ four ferrite slabs at equivalent positions in the waveguide cross section, this isolator utilizes only one in view of the magnet geometry. Also, only a small part of the slab is at resonance at any given frequency. The situation may be improved, however, by reducing the waveguide height and increasing the ferrite thickness. This would again call for some empirical development because both of these measures are known to alter the RF and dc magnetic field distributions.

The isolator type described here may offer advantages in applications calling for isolation over a broad band characterized by low insertion loss and a high ratio of reverse-to-forward attenuation. While there are limits to the ratio and bandwidth that can be achieved simultaneously with a particular ferrite material, they are considerably higher than predicted by the theory assuming uniform dc fields.

Acknowledgment

The author is indebted to J. A. Weiss for various suggestions during the course of this work.

Appendix

Calculation of the Forward and Reverse Attenuation Constants

The imaginary part of the external susceptibility can be expressed as

$$\chi_{++}'' = \frac{\omega_m T^{-1}}{(\omega_e - \omega)^2 + T^{-2}} \tag{17}$$

$$\chi_{--}'' = \frac{\omega_m T^{-1}}{(\omega_e + \omega)^2 + T^{-2}} \tag{18}$$

for the positive and negative circular polarization, respectively. It is assumed that $N_y = N_x = 0$, and

$$\omega_e \equiv \gamma H_e = \gamma(H_z - N_z 4\pi M). \tag{19}$$

The RF magnetic field in the empty guide at the site of the ferrite slab can be expressed in terms of circular polarization as

$$|h_+| = \tfrac{1}{2}(d\cos\theta + \sin\theta) \tag{20}$$

$$|h_-| = \tfrac{1}{2}(d\cos\theta - \sin\theta) \tag{21}$$

where the RF magnetic field at $x=0$ is taken as unity, and

$$d \equiv \sqrt{\left(\frac{\omega}{\omega_c}\right)^2 - 1}. \tag{22}$$

Then (3) and (4) are

$$\alpha_+ = \frac{2t}{Sd}\int [|h_+|^2\chi_{++}'' + |h_-|^2\chi_{--}'']dx$$

$$\approx \frac{2t}{Sd}\int |h_+|^2\chi_{++}''dx \tag{23}$$

$$\alpha_- = \frac{2t}{Sd}\int [|h_-|^2\chi_{++}'' + |h_+|^2\chi_{--}'']dx. \tag{24}$$

In (19), we assume the effective field H_e changes along the x axis as

$$\omega_e = \omega_0 \sec\theta.$$

For a given frequency ω_1, the field at $\theta = \theta_1$ where $|h_-| = 0$ is tuned to resonance, *i.e.*,

$$\omega_1 = \omega_0 \sec\theta_1. \tag{25}$$

χ'''s and h's are expanded in Taylor series about $\theta = \theta_1$ and retain only the first nonvanishing term in the expansion. These expressions are then substituted into (23) and (24), and the result of the integration assuming the ferrite slab extending from $\theta = \pi/6$ to $\theta = \pi/3$ radian becomes

$$\alpha_- \approx \frac{t}{S}\cdot\left[\frac{\omega_m}{2\omega_0^2 T}\cdot\left(p - 2q\frac{f}{f'}\right)\cdot\frac{\pi}{6}\cdot\frac{1}{(2f - f'\Delta\theta_1)^2} + \frac{\omega_m q}{2\omega_0^2 T f'^2}\cdot\ln\frac{2f + f'\left(\frac{\pi}{12} - \Delta\theta_1\right)}{2f - f'\left(\frac{\pi}{12} + \Delta\theta_1\right)} + \frac{\omega_m n}{2\omega_0^2 T f'^2}\cdot\frac{\pi}{6}\right] \tag{26}$$

$$\alpha_+ \approx \frac{t}{S}\cdot\left[\frac{\omega_m p\pi}{2\omega_0 f'}\left\{1 - \frac{1}{6\omega_0 f' T}\cdot\frac{1}{\left(\frac{\pi}{12}\right)^2 - (\Delta\theta_1)^2}\right\} + \frac{\omega_m q}{4\omega_0^2 f'^2 T}\cdot\ln\left\{1 - \frac{\frac{\pi}{3}\Delta\theta_1}{\left(\frac{\pi}{12} + \Delta\theta_1\right)^2 + \frac{1}{\omega_1^2 f'^2 T^2}}\right\}\right] \tag{27}$$

where

$\Delta\theta_1 = \theta_1 - \pi/4,$
$f = \sec\theta_1,\ f' = \sec\theta_1 \cdot \tan\theta_1$
$p = 4\sin^2\theta_1,\ q = 4\tan\theta_1 \cdot \cos 2\theta_1$
$n = \sec^2\theta_1.$

Eqs. (26) and (27) are plotted in Fig. 2.